Robotics II

July 15, 2013

Consider a planar 3R robot on a horizontal plane, having link lengths ℓ_1, ℓ_2, and ℓ_3. Figure 1 shows three different situations. In each of them, the robot is subject to:

- a *single unknown* external force $\boldsymbol{F}_i = \begin{pmatrix} F_{xi} & F_{yi} \end{pmatrix}^T$, of arbitrary direction and intensity in the plane, applied to a point of the first, second, or third link, at a (*possibly, unknown*) distance ℓ_{ci} from the link i base (with $i = 1, 2, 3$, respectively);
- an associated *known* (measured, or computed by a controller) joint torque $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix}^T$ that keeps the robot in a static equilibrium $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$, as *measured* by encoders.

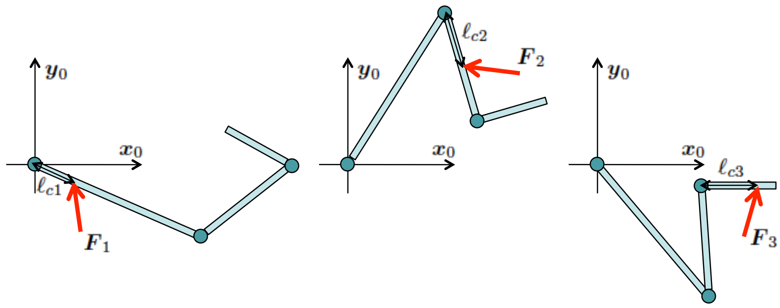

Figure 1: A planar 3R robot subject to an unknown external force $\boldsymbol{F}_i$ applied to the first ($i = 1$, left), second ($i = 2$, center), or third link ($i = 3$, right)

Analyze **for each** of the above three generic situations *if* and *how:*

1. one can identify which link is subject to the external force;
2. knowing the distance ℓ_{ci} of the application point along the link (which can be obtained by means of an external camera, a Kinect, or using distributed tactile sensing), one can *completely* estimate $\boldsymbol{F}_i$, e.g., its direction and intensity;
3. one can estimate $\boldsymbol{F}_i$ *and* the distance $\ell_{c,i}$ of the application point along the link, without any external/extra sensing;
4. the presence of gravity (with the robot being in a vertical plane) makes a difference for the above problems.

Verify the analysis for a 3R robot in a horizontal plane, with the numerical data

$$\ell_1 = 0.5 \text{ [m]}, \quad \ell_2 = 0.3 \text{ [m]}, \quad \ell_3 = 0.2 \text{ [m]}, \qquad \boldsymbol{q} = \begin{pmatrix} 45^\circ & -90^\circ & 60^\circ \end{pmatrix}^T \text{ (in D-H convention)},$$

and for the following two cases of known equilibrium joint torques:

$$a)\ \boldsymbol{\tau}_a = \begin{pmatrix} -0.25 & -0.75 & 0 \end{pmatrix}^T \text{ [Nm]}, \qquad b)\ \boldsymbol{\tau}_b = \begin{pmatrix} 0.7585 & -0.2995 & 0.2000 \end{pmatrix}^T \text{ [Nm]}.$$

Estimate the applied external force in case *a)* and *b)*, respectively. Following the outcome of your analysis, try to work without any a priori knowledge of the application point of the external force. If a case turns out to be under-determined, choose the application point at the *link midpoint.*

[120 minutes; open books]

Solution

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In the absence of gravity, the joint torque that balances at an equilibrium $\boldsymbol{q}$ a force applied to a point along the robot structure having position $\boldsymbol{p}_c = \boldsymbol{f}_c(\boldsymbol{q})$ is given by

$$\boldsymbol{\tau} = -\left(\frac{\partial \boldsymbol{f}_c}{\partial \boldsymbol{q}}\right)^T \boldsymbol{F} = -\boldsymbol{J}_c^T(\boldsymbol{q})\,\boldsymbol{F}.$$

Planar forces $\boldsymbol{F} \in \mathbb{R}^2$ are expressed here in the base frame $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. From the position of the contact point $\boldsymbol{p}_c \in \mathbb{R}^2$, we can derive the *contact Jacobian* associated to a force $\boldsymbol{F}_i$ acting on link i at a distance ℓ_{ci} from its base, for $i = 1, 2, 3$ (three cases).

For a force $\boldsymbol{F}_1$ acting on the first link, with $\ell_{c1} \in (0, \ell_1]$:

$$\boldsymbol{J}_{c1} = \begin{pmatrix} -\ell_{c1}\sin q_1 & 0 & 0 \\ \ell_{c1}\cos q_1 & 0 & 0 \end{pmatrix}.$$

For a force $\boldsymbol{F}_2$ acting on the second link, with $\ell_{c2} \in (0, \ell_2]$:

$$\boldsymbol{J}_{c2} = \begin{pmatrix} -\ell_1\sin q_1 - \ell_{c2}\sin(q_1+q_2) & -\ell_{c2}\sin(q_1+q_2) & 0 \\ \ell_1\cos q_1 + \ell_{c2}\cos(q_1+q_2) & \ell_{c2}\cos(q_1+q_2) & 0 \end{pmatrix}.$$

For a force $\boldsymbol{F}_3$ acting on the third link, with $\ell_{c3} \in (0, \ell_3]$:

$$\boldsymbol{J}_{c3} = \begin{pmatrix} -\ell_1 s_1 - \ell_2 s_{12} - \ell_{c3} s_{123} & -\ell_2 s_{12} - \ell_{c3} s_{123} & -\ell_{c3} s_{123} \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_{c3} c_{123} & \ell_2 c_{12} + \ell_{c3} c_{123} & \ell_{c3} c_{123} \end{pmatrix},$$

where we used the usual compact notation, e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$.

In order to have a better insight on the contact forces that are felt at the robot joints, it is convenient to express these forces in the local frame attached to each link, i.e., according to their tangential and normal components w.r.t. the geometric link. Since we are dealing with a purely planar problem, the planar rotation matrices of interest are

$${}^0\boldsymbol{R}_1 = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}, \quad {}^0\boldsymbol{R}_2 = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}, \quad {}^0\boldsymbol{R}_3 = \begin{pmatrix} c_{123} & -s_{123} \\ s_{123} & c_{123} \end{pmatrix}.$$

we have

$$\boldsymbol{F}_i = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{F}_i \quad \Rightarrow \quad \boldsymbol{\tau}_i = -\boldsymbol{J}_{ci}^T(\boldsymbol{q})\,\boldsymbol{F}_i = -\boldsymbol{J}_{ci}^T(\boldsymbol{q}){}^0\boldsymbol{R}_i\,{}^i\boldsymbol{F}_i = -\left({}^0\boldsymbol{R}_i^T\boldsymbol{J}_{ci}(\boldsymbol{q})\right)^T {}^i\boldsymbol{F}_i = -{}^i\boldsymbol{J}_{ci}^T(\boldsymbol{q})\,{}^i\boldsymbol{F}_i.$$

Once expressed in the local link frame, the contact Jacobians ${}^i\boldsymbol{J}_{ci}(\boldsymbol{q})$ for the considered cases are:

$${}^1\boldsymbol{J}_{c1} = {}^0\boldsymbol{R}_1^T\boldsymbol{J}_{c1}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ \ell_{c1} & 0 & 0 \end{pmatrix}, \qquad \textit{always} \text{ of rank } 1;$$

$${}^2\boldsymbol{J}_{c2} = {}^0\boldsymbol{R}_2^T\boldsymbol{J}_{c2}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 s_2 & 0 & 0 \\ \ell_1 c_2 + \ell_{c2} & \ell_{c2} & 0 \end{pmatrix}, \qquad \text{of rank 2 } \textit{if and only if } |s_2| \neq 0;$$

$${}^3\boldsymbol{J}_{c3} = {}^0\boldsymbol{R}_3^T\boldsymbol{J}_{c3}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 s_{23} + \ell_2 s_3 & \ell_2 s_3 & 0 \\ \ell_1 c_{23} + \ell_2 c_3 + \ell_{c3} & \ell_2 c_3 + \ell_{c3} & \ell_{c3} \end{pmatrix}, \text{ of rank 2 } \textit{if and only if } s_2^2 + s_3^2 \neq 0.$$

Therefore, we have the following explicit equations at steady state.

For a force ${}^1\boldsymbol{F}_1 = \begin{pmatrix} {}^1F_{1x} & {}^1F_{1y} \end{pmatrix}^T$ acting on the first link:

$$\begin{aligned} \tau_{1,1} &= -\ell_{c1}\,{}^1F_{1y}, \\ \tau_{1,2} &= 0, \\ \tau_{1,3} &= 0. \end{aligned} \tag{1}$$

For a force ${}^2\boldsymbol{F}_2 = \begin{pmatrix} {}^2F_{2x} & {}^2F_{2y} \end{pmatrix}^T$ acting on the second link:

$$\begin{aligned} \tau_{2,1} &= -\ell_1 s_2 {}^2F_{2x} - (\ell_1 c_2 + \ell_{c2})\,{}^2F_{2y} = -\ell_1\left(s_2\,{}^2F_{2x} + c_2\,{}^2F_{2y}\right) + \tau_{2,2}, \\ \tau_{2,2} &= -\ell_{c2}\,{}^2F_{2y}, \\ \tau_{2,3} &= 0. \end{aligned} \tag{2}$$

For a force ${}^3\boldsymbol{F}_3 = \begin{pmatrix} {}^3F_{3x} & {}^3F_{3y} \end{pmatrix}^T$ acting on the third link:

$$\begin{aligned} \tau_{3,1} &= -\left(\ell_1 s_{23} + \ell_2 s_3\right){}^3F_{3x} - \left(\ell_1 c_{23} + \ell_2 c_3 + \ell_{c3}\right){}^3F_{3y} \\ &= -\ell_1\left(s_{23}\,{}^3F_{3x} + c_{23}\,{}^3F_{3y}\right) + \tau_{3,2}, \\ \tau_{3,2} &= -\left(\ell_2 s_3\,{}^3F_{3x} + (\ell_3 c_3 + \ell_{c3})\,{}^3F_{3y}\right) = -\ell_2\left(s_3\,{}^3F_{3x} + c_3\,{}^3F_{3y}\right) + \tau_{3,3}, \\ \tau_{3,3} &= -\ell_{c3}\,{}^3F_{3y}. \end{aligned} \tag{3}$$

Inner recursions from the outer to the inner joints have been used to simplify the expressions. Based on the above, the following series of observations can be made.

1. Identification of which link is subject to the contact force is made using the components of the joint torque vector $\boldsymbol{\tau}_i$, based on the following cascaded (generic) rule:

$$\begin{aligned} \tau_{i,3} = \tau_{i,2} = 0 \quad &\Rightarrow \quad i = 1, \text{ link 1 is involved} \\ \tau_{i,3} = 0, \tau_{i,2} \neq 0 \quad &\Rightarrow \quad i = 2, \text{ link 2 is involved} \\ \text{else} \quad &\Rightarrow \quad i = 3, \text{ link 3 is involved.} \end{aligned}$$

   Note that, if a force is applied at the base of link i (corresponding to the exact location of joint i), it is then attributed to the tip of the previous link $i-1$ (with $\ell_{c,i-1} = \ell_{i-1}$), since $\ell_{ci} \neq 0$ by definition.

2. Knowledge of the joint torque vector $\boldsymbol{\tau}_i$ at an equilibrium configuration $\boldsymbol{q}$ is obtained in one of the following alternative, but equivalent ways (however, note the signs!):
   - from the static measurement by a joint torque sensor, $\boldsymbol{\tau}_m = -\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{\tau}_i$;
   - as the steady-state value of the residual vector $\boldsymbol{r}$ generated in response to a collision with a constant contact force $\boldsymbol{F}_c$, $\boldsymbol{r} = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = -\boldsymbol{\tau}_i$;
   - as the steady-state output of a feedback controller, e.g., with a regulation law designed for a desired position $\boldsymbol{q}_d$ and including a proportional term, $\boldsymbol{\tau}_c = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) = -\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{\tau}_i$.

3. Since the rank of matrix $\boldsymbol{J}_{c1}$ (and of ${}^1\boldsymbol{J}_{c1}$) is always one, we can *never* estimate completely a contact force acting on link 1. As intuition suggests, from eqs. (1) we can only estimate the force component ${}^1F_{1y}$ that is normal to the link, once we know its application point, as

$$ {}^1F_{1y} = -\frac{\tau_{1,1}}{\ell_{c1}}, $$

whereas ${}^1F_{1x}$ can be arbitrary and is balanced by the internal reaction force of the link structure.

4. When matrix $\boldsymbol{J}_{c2}$ (${}^2\boldsymbol{J}_{c2}$) has full rank, by using eqs. (2) we can estimate completely the contact force $\boldsymbol{F}_2$ acting on link 2, *provided* that ℓ_{c2} is known, as

$$^2F_{2y} = -\frac{\tau_{2,2}}{\ell_{c2}}, \qquad ^2F_{2x} = -\frac{1}{\ell_1 s_2}\left(\tau_{2,1} - (\ell_1 c_2 + \ell_{c2})\,\frac{\tau_{2,2}}{\ell_{c2}}\right).$$

   To recover the expression of the contact force in the base frame, we use $\boldsymbol{F}_2 = {}^0\boldsymbol{R}_2\,{}^2\boldsymbol{F}_2$.

5. When the second link is aligned with the first one ($s_2 = 0$), $\boldsymbol{J}_{c2}$ loses rank and we recover a similar situation to that in item 3, namely ${}^2F_{2x}$ cannot be estimated. However, in this case the two data $\tau_{2,1}$ and $\tau_{2,2}$ can be used to estimate *both* ${}^2F_{2y}$ and ℓ_{c2} as

$$^2F_{2y} = \pm\,\frac{\tau_{2,1} - \tau_{2,2}}{\ell_1}, \qquad \ell_{c2} = -\frac{\tau_{2,2}}{^2F_{2y}},$$

   where the sign '$-$' corresponds to $q_2 = 0$ and the sign '$+$' to $q_2 = \pi$.

6. When matrix ${}^3\boldsymbol{J}_{c3}$ (or $\boldsymbol{J}_{c3}$) has full rank and ℓ_{c3} is known, by using eqs. (3) we can estimate completely the contact force $\boldsymbol{F}_3$ acting on link 3 as

$$\begin{aligned} ^3F_{3y} &= -\frac{\tau_{3,3}}{\ell_{c3}}, \\ ^3F_{3x} &= -\frac{1}{\ell_2 s_3}\left(\tau_{3,2} - (\ell_2 c_3 + \ell_{c3})\,\frac{\tau_{3,3}}{\ell_{c3}}\right), \qquad \text{if } s_3 \neq 0 \\ \text{or} \qquad ^3F_{3x} &= -\frac{1}{\ell_2 s_{23}}\left(\tau_{3,1} - \tau_{3,2} - \ell_1 c_{23}\,\frac{\tau_{3,3}}{\ell_{c3}}\right), \qquad \text{if } s_3 = 0, \text{ but } s_{23} = \pm\, s_2 \neq 0. \end{aligned}$$

   To recover the expression of the contact force in the base frame, we use $\boldsymbol{F}_3 = {}^0\boldsymbol{R}_3\,{}^3\boldsymbol{F}_3$. When the robot arm is fully stretched or folded, we can proceed as in item 5 and identify from $\boldsymbol{\tau}_3$ both ${}^3F_{3y}$ and ℓ_{c3}, but not ${}^3F_{3x}$.

7. One interesting feature of having three independent information from the joint torque vector in case of contact on the third link, is that the estimation of the force $\boldsymbol{F}_3$ can be performed even without knowing ℓ_{c3}. In fact, the value ℓ_{c3} has disappeared in the recursive expressions of the first two equations in (3). Therefore, we can evaluate

$$^3\boldsymbol{F}_3 = \begin{pmatrix} ^3F_{3x} \\ ^3F_{3y} \end{pmatrix} = -\begin{pmatrix} \ell_1 s_{23} & \ell_1 c_{23} \\ \ell_2 s_3 & \ell_2 c_3 \end{pmatrix}^{-1} \begin{pmatrix} \tau_{3,1} - \tau_{3,2} \\ \tau_{3,2} - \tau_{3,3} \end{pmatrix},$$

   provided that $s_2 \neq 0$ holds (for the invertibility of the coefficient matrix), and then complete the analysis by using the third equation in (3)

$$\ell_{c3} = -\frac{\tau_{3,3}}{^3F_{3y}}.$$

8. The results in the above items 3, 4, and 6 can all be obtained by using the *pseudoinverse* of the associated contact Jacobian, independently from its rank,

$$\hat{\boldsymbol{F}}_i = -\left(\boldsymbol{J}_{ci}^T(\boldsymbol{q})\right)^{\#}\boldsymbol{\tau}_i \qquad \text{or} \qquad {}^i\hat{\boldsymbol{F}}_i = -\left({}^i\boldsymbol{J}_{ci}^T(\boldsymbol{q})\right)^{\#}\boldsymbol{\tau}_i, \qquad \text{for } i = 1, 2, 3.$$

The 'hat' has been added to express the fact that the estimation may not be complete (e.g., in the contact on the first link, or in the other two cases when the contact Jacobian loses rank). On the other hand, the estimation result in item 7 is obtained by direct inspection of the equations.

9. The presence of a gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamics does not change the picture substantially. The main difference is that the joint torque $\boldsymbol{\tau}$ should be replaced in all above formulas by $\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q})$, since at the equilibrium

$$\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}.$$

For the two assigned numerical problems, we have the following results. We note first that the given robot configuration $\bar{\boldsymbol{q}} = \begin{pmatrix} \pi/4 & -\pi/2 & \pi/3 \end{pmatrix}^T$ [rad] does not lead to singularity problems.

a) Contact occurs on the second link (see item 1) and eqs. (2) apply. Since knowledge of the application point is necessary in this case, we set $\ell_{c2} = \ell_2/2 = 0.15$ [m]. The contact Jacobian is then

$$\boldsymbol{J}_{c2}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.2475 & 0.1061 & 0 \\ 0.4596 & 0.1061 & 0 \end{pmatrix}.$$

Thus,

$$\boldsymbol{F}_2 = -\left(\boldsymbol{J}_{c2}^T(\bar{\boldsymbol{q}})\right)^{\#}\boldsymbol{\tau}_a = \begin{pmatrix} 4.242 \\ 2.8284 \end{pmatrix}, \quad {}^2\boldsymbol{F}_2 = {}^0\boldsymbol{R}_2^T(\bar{\boldsymbol{q}})\boldsymbol{F}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \text{both expressed in [N].}$$

b) Contact occurs on the third link and eqs. (3) apply. Solving for contact force and distance of its application point from the link base yields

$$\boldsymbol{F}_3 = \begin{pmatrix} 2.6736 \\ -0.3189 \end{pmatrix}, \quad {}^3\boldsymbol{F}_3 = \begin{pmatrix} 2.5 \\ -1 \end{pmatrix}, \quad \text{both expressed in [N];} \quad \ell_{c3} = 0.2\,\text{[m] (i.e., at the tip).}$$

Using the computed ℓ_{c3}, the resulting contact Jacobian is then

$$\boldsymbol{J}_{c3}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.1932 & 0.1604 & -0.0518 \\ 0.7589 & 0.4053 & 0.1932 \end{pmatrix}.$$

Finally, one can verify that $\boldsymbol{F}_3 = -\left(\boldsymbol{J}_{c3}^T(\bar{\boldsymbol{q}})\right)^{\#}\boldsymbol{\tau}_b$.

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